NONLINEAR THERMODYNAMIC FORMALISM THROUGH THE LENS OF ROTATION THEORY

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ABSTRACT. We discuss the connection between the nonlinear thermodynamic formalism recently introduced by Buzzi, Kloeckner, and Leplaideur and the theory of generalized rotation sets which goes back to Poincaré.

1. INTRODUCTION

Thermodynamic formalism was largely developed by Sinai, Ruelle and Bowen in 1970s and has its origins in statistical mechanics of lattice spin systems. Since then the theory has aided significant advances in the study of various dynamical systems and even in more general setups. Recently, Leplaideur and Watbled applied the tools from thermodynamic formalism to the Curie-Weiss mean-field theory with a new twist: they used a variant of the pressure where the energy functional is quadratic. Inspired by this result, Buzzi, Kloeckner, and Leplaideur initiated an effort to broaden the thermodynamical approach by considering an arbitrary function of free energies.

The primary goal of this note is to show that the appropriate framework for the nonlinear thermodynamic formalism is rotation theory. This theory has its origins in the work of Poincaré on rotation numbers for circle homeomorphisms, but it has now invaded many branches of dynamics, including symbolic systems, billiards, and continuous interval maps. We establish a formula which allows to compute the nonlinear pressure using the localized entropy function introduced by Misiurewicz in 1989 and subsequently studied by Jenkinson, Lopes, Wolf and others. This places numerous products and insights of rotation theory at our disposal, which enables us to give simpler proofs of the results in [2] and [1]. Although we obtain several useful facts concerning the pressure in the nonlinear settings, we believe that the main value of this work lies in connecting the newest direction in thermodynamic formalism to the century old theory of rotation sets. Once this connection is made, many results about the structure of the nonlinear equilibrium states obtained in [1, 2] can be deduced from [5, 12].

Key words and phrases. topological pressure, variational principle, equilibrium states, thermodynamic formalism, rotation sets.

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We consider a continuous dynamical system $T : X \to X$ where X is a compact metric space. In [2] the authors define the *nonlinear pressure* of an invariant probability measure μ to be

$$\Pi^F(T, \phi_1, ..., \phi_m, \mu) = h_\mu(T) + F\left(\int \phi_1 d\mu, ..., \int \phi_m d\mu\right), \qquad (1)$$

where $\phi_1, ..., \phi_m$ are fixed real valued potentials on X and the *nonlinearity* $F : \mathbb{R}^m \mapsto \mathbb{R}$ is a continuous multivariable function. They further introduce a topological analog of nonlinear pressure by adopting Bowen-Dinaburg approach and show that it coincides with the supremum of the nonlinear pressure of the measures, hence establishing a nonlinear variational principle. One of the most interesting outcomes of the theory is a characterization of the nonlinear equilibrium measures as classical equilibrium states for some linear combinations of the potential ϕ_i. An application of this approach with $F(z) = z^2$ leads to a better understanding of the properties of Gibbs measures for generalized Curie-Weiss model [15], which provided an inspiration for the nonlinear thermodynamic formalism theory.

In this note we demonstrate that the nonlinear topological pressure maximizes the sum of the nonlinearity F and the localized topological entropy over the rotation set of the potential $\Phi = (\phi_1, ..., \phi_m)$. This allows us to use the theory of rotation sets and localized equilibrium states developed over the years in [3, 5, 11, 12] and most recently in [4]. We obtain a nonlinear version of the variational principle for a vector-valued potential. One interesting outcome of our method of proof is a surprising fact that contrary to the classical case the upper limit in the topological definition of the nonlinear pressure cannot be replaced by the lower limit (see Example 2. Another is that even in the case when $m = 1$ the nonlinear variational principle does not hold in restriction to ergodic measures, which answers the question in [2, Question 1.6].

2. Preliminaries

2.1. **Classical Thermodynamic Formalism.** Let (X, d) be a compact metric space and $T : X \to X$ be a continuous map . We denote by $\mathcal{M}$ the set of all Borell T-invariant probability measures on X endowed with weak* topology and by $\mathcal{M}^e \subset \mathcal{M}$ the subset of ergodic measures. For $n \in \mathbb{N}$ we define a new metric d_n on X by $d_n(x, y) = \max_{k=0,...,n-1} d(T^k(x), T^k(y))$. Note that d_n is a metric (called Bowen metric) that induces the same topology on X as d. We denote by $B_n(x, \rho)$ the open ball of radius ρ centered at $x \in X$ with respect to the d_n metric. We say that $E \subset X$ is (n, ε)-separated if for all $x, y \in E$ with $x \neq y$ we have $d_n(x, y) \geq \varepsilon$.

Consider the space $C(X, \mathbb{R})$ of continuous real valued functions on X. Given a potential $\phi \in C(X, \mathbb{R})$ we denote by $S_n\phi(x)$ the Birkhoff sum at x of length n with respect to ϕ, i.e. $S_n\phi(x) = \sum_{k=0}^{n-1} \phi(T^k(x))$. For $n \in \mathbb{N}$ and

$\varepsilon > 0$ let

$$Z_\phi(n,\varepsilon) = \sup\left\{\sum_{x\in E} e^{S_n\phi(x)} \,:\, E\subset X \text{ is } (n,\varepsilon)\text{-separated}\right\}. \tag{2}$$

The *topological pressure* with respect to the dynamical system (X,T) is a mapping $P_{\rm top}(T,\cdot)\colon C(X,\mathbb{R})\to\mathbb{R}\cup\{\infty\}$ defined by

$$P_{\rm top}(T,\phi) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log Z_\phi(n,\varepsilon). \tag{3}$$

The *topological entropy* of T is defined by $h_{\rm top}(T) = P_{\rm top}(T,0)$. Hence,

$$h_{\rm top}(T) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\log \operatorname{card} E_n(\varepsilon), \tag{4}$$

where $E_n(\varepsilon)$ is a maximal (with respect to the inclusion) (n,ε)-separated set. Note that using lim inf instead of lim sup in the above leads to the equivalent definitions of the pressure and the entropy respectively. We simply write $P_{\rm top}(\phi)$ and $h_{\rm top}$ if there is no confusion about T. The topological pressure is finite if and only if the topological entropy of T is finite. We use $h_{\rm top}(T)<\infty$ as a standing assumption in this paper. The topological pressure satisfies the well-known variational principle

$$P_{\rm top}(\phi) = \sup_{\mu\in\mathcal{M}}\left\{h_\mu(T) + \int_X \phi\, d\mu\right\}. \tag{5}$$

Here $h_\mu(T)$ denotes the measure-theoretic entropy of T with respect to μ (see [18] for details). It is a straight forward conclusion that the supremum in (5) can be replaced by the supremum taken only over all $\mu\in\mathcal{M}^e$.

2.2. **Rotation Theory.** Rotation theory in its general form considers a dynamical system and an accompanying potential function on the phase space. Then the rotation set is defined as the collection of possible limits arising from ergodic averages. Precisely, for a continuous map $T: X\to X$ on a compact metric space X and a continuous potential $\Phi = (\phi_1,...,\phi_m): X\to\mathbb{R}^m$ the *pointwise rotation set* of Φ is

$$\mathrm{Rot}_{Pt}(\Phi) = \left\{\lim_{l\to\infty}\frac{1}{n_l}\sum_{i=1}^{n_l}\Phi(T^i(x_l)) : (x_l)\subset X,\, n_l\to\infty\right\}. \tag{6}$$

The terminology and ideas behind the rotation theory come from Poincaré's rotation numbers for circle maps and the first generalizations of this classical set up for homeomorphisms of the 2-dimensional torus. In these cases the potential is taken to be the displacement function of the homeomorphism, and the rotation set is used to describe the asymptotic motion of orbits of the homeomorphism. The magnitude of a vector in the rotation set gives the speed of motion, and its direction gives a homology class which best approximates the motion. A transition from the rotation theory on the torus to the abstract situation where we have a general dynamical system and an arbitrary continuous potential was done by Geller and Misiurewicz in [3].

The pointwise rotation set is difficult to work with a priori, so Geller and Misiurewicz introduced the now standard definition of the *(generalised) rotation set*

$$\mathrm{Rot}(\Phi) = \left\{ \left(\int \phi_1 d\mu, ..., \int \phi_m d\mu \right) \in \mathbb{R}^m \, : \mu \in \mathcal{M} \right\}. \tag{7}$$

We use the notation $\mathrm{rv}_\Phi(\mu) = \left(\int \phi_1 \, d\mu, \dots, \int \phi_m \, d\mu\right)$ for the *rotation vector* of the measure μ. We note that $\mathrm{Rot}(\Phi)$ is a compact convex subset of $\mathbb{R}^m$, while the pointwise rotation set $\mathrm{Rot}_{Pt}(\Phi)$ is compact, but not necessarily convex. We have $\mathrm{Rot}_{Pt}(\Phi) \subset \mathrm{Rot}(\Phi)$ where the inclusion may be strict, and $\mathrm{conv}\mathrm{Rot}_{Pt}(\Phi) = \mathrm{Rot}(\Phi)$, see [13] for details.

For each point w in a rotation set we can associate a local version of the topological entropy. It is computed in terms of the exponential growth rate of the cardinality of maximal (n, ε)-separated sets of points whose Birkhoff averages are "close" to w (see [11]). Precisely, let $w \in \mathbb{R}^m$, $n \in \mathbb{N}$ and $\varepsilon, r > 0$. To make a distinction from the balls in the Bowen metric, we denote by $D(w, r)$ the open ball of radius r centered at $w \in \mathbb{R}^m$ with respect to the Euclidian metric. A set $E \subset X$ is said to be a (n, ε, w, r)-set if E is (n, ε)-separated and $\frac{1}{n} S_n\Phi(x) = \left(\frac{1}{n} S_n\phi_1(x), \dots, \frac{1}{n} S_n\phi_m(x)\right) \in D(w, r)$ for all $x \in E$. For all $n \in \mathbb{N}$ and $\varepsilon, r > 0$ let $E_n(\varepsilon, w, r)$ be a maximal (with respect to the inclusion) (n, ε, w, r)-set. We define

$$h(w, r) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{card}\, E_n(\varepsilon, w, r), \tag{8}$$

where we interpret $\log 0 = -\infty$. Then the *localized topological entropy* of w is

$$h(w) = \lim_{r \to 0} h(w, r). \tag{9}$$

Analogously to the case of $h_{\mathrm{top}}(T)$ one can show that $h(w)$ does not depend on the choice of the (n, ε, w, r)-sets $E_n(\varepsilon, w, r)$.

It follows directly from the definition that the map $w \mapsto h(w)$ is upper semi-continuous and bounded above by $h_{\mathrm{top}}(T)$. In order for $h(w) \geq 0$ we need $D(w, r)$ to contain statistical averages with respect to Φ for infinitely many n and arbitrarily small r. The set of points in $\mathbb{R}^m$ which satisfy this property is exactly the pointwise rotation set of Φ.

3. Nonlinear Pressure

We continue using the notations from Section 2. We fix a continuous nonlinearity $F : \mathbb{R}^m \to \mathbb{R}$ and a vector valued continuous potential $\Phi = (\phi_1, ..., \phi_m) : X \to \mathbb{R}^m$. Recall that $\frac{1}{n} S_n\Phi(x)$ denotes the m-dimensional Birkhoff average at x of length n with respect to Φ, where $S_n\Phi(x) = (S_n\phi_1(x), \dots, S_n\phi_m(x))$ and $S_n\phi_i(x) = \sum_{k=0}^{n-1} \phi_i(T^k(x))$. For $n \in \mathbb{N}$ and

$\varepsilon > 0$ let

$$\mathcal{Z}^F(n,\varepsilon) = \sup\left\{\sum_{x\in E} e^{nF\left(\frac{1}{n}S_n\Phi(x)\right)} : E \text{ is } (n,\varepsilon)\text{-separated set}\right\}. \tag{10}$$

We define the *nonlinear topological pressure* of Φ with respect to F as

$$\Pi^F_{\text{top}}(\Phi) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log \mathcal{Z}^F(n,\varepsilon) \tag{11}$$

We note that in [2] Buzzi, Kloeckner, and Leplaideur defined the nonlinear pressure using the optimal covers of X by ε-balls in the d_n metric rather than (n,ε)-separated sets. It is a standard argument to show that

$$\mathcal{Z}^F(n,2\varepsilon) \le \inf\left\{\sum_{x\in E} e^{nF\left(\frac{1}{n}S_n\Phi(x)\right)} : \bigcup_{x\in E} B_n(x,\varepsilon) = X\right\} \le \mathcal{Z}^F(n,\varepsilon).$$

Therefore, the nonlinear pressure given by (11) coincides with the one introduced in [2].

In the next theorem we establish a connection between the nonlinear topological pressure and the localized topological entropy.

Theorem 1. *Let $T : X \to X$ be a continuous map on a compact metric space X and $F : \mathbb{R}^m \to \mathbb{R}$ be a continuous nonlinearity. Then for any continuous potential $\Phi : X \to \mathbb{R}^m$*

$$\Pi^F_{\text{top}}(\Phi) = \sup_{w\in \mathrm{Rot}_{Pt}(\Phi)}\{h(w) + F(w)\}$$

Proof. The lower bound on $\Pi^F_{\text{top}}(\Phi)$ follows in a straightforward way from the continuity of F and the definitions of nonlinear pressure and localized entropy. To see this, fix some $w \in \mathrm{Rot}_{Pt}(\Phi)$ and $\eta > 0$. Since F is continuous at w, we can find $r_0 > 0$ such that for any $v \in D(w, r_0)$ we have $|F(v) - F(w)| < \eta$. Note that every (n,ε,w,r)-set is, in particular, (n,ε)-separated set. Hence, whenever $r < r_0$ for each $n \in \mathbb{N}$ and $\varepsilon > 0$ we have

$$\begin{aligned}\mathcal{Z}^F(n,\varepsilon) &= \sup\left\{\sum_{x\in E} e^{nF\left(\frac{1}{n}S_n\Phi(x)\right)} : E \text{ is } (n,\varepsilon)\text{-separated set}\right\}\\ &\ge \sup\left\{\sum_{x\in E} e^{nF\left(\frac{1}{n}S_n\Phi(x)\right)} : E \text{ is } (n,\varepsilon,w,r)\text{-set}\right\}\\ &= e^{n(F(w)-\eta)}\mathrm{card}E_n(\varepsilon,w,r),\end{aligned}$$

where $E_n(\varepsilon,w,r)$ is a maximal (n,ε,w,r)-set. Therefore,

$$\Pi^F_{\text{top}}(\Phi) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log \mathcal{Z}^F(n,\varepsilon) \ge F(w) - \eta + h(w,r). \tag{12}$$

Letting $r \to 0$ gives $\Pi^F_{\rm top}(\Phi) \geq F(w) - \eta + h(w)$. Since $\eta > 0$ and $w \in \mathrm{Rot}_{Pt}(\Phi)$ were arbitrary, we obtain the desired lower bound

$$\Pi^F_{\rm top}(\Phi) \geq \sup_{w \in \mathrm{Rot}_{Pt}(\Phi)} \{h(w) + F(w)\}$$

Now we turn our attention to the opposite inequality. Let $\eta > 0$ be arbitrary. Since F is uniformly continuous on the convex hull of the image of Φ, there is $r > 0$ such that for any $v, w \in \mathrm{conv}\Phi(X)$ with $\|v - w\| \leq 2r$ we have $|F(v) - F(w)| < \frac{\eta}{3}$. We fix a finite cover of $\mathrm{Rot}(\Phi)$ by balls of radius r centered at $w_i \in \mathrm{Rot}(\Phi)$ where $i = 1, ..., k$. Then for any $n \in \mathbb{N}$ and $\varepsilon > 0$ we have

$$\begin{aligned}
\mathcal{Z}^F(n, \varepsilon) &= \sup \left\{ \sum_{x \in E} e^{nF\left(\frac{1}{n} S_n \Phi(x)\right)} : E \text{ is } (n, \varepsilon)\text{-separated set} \right\} \\
&\leq \sum_{i=1}^{k} \sup \left\{ \sum_{x \in E_i} e^{nF\left(\frac{1}{n} S_n \Phi(x)\right)} : E_i \text{ is } (n, \varepsilon, w_i, r)\text{-set} \right\} \\
&\leq k \max_{1 \leq i \leq k} \left\{ e^{n\left(F(w_i) - \frac{\eta}{3}\right)} \mathrm{card} E_n(\varepsilon, w_i, r) \right\}
\end{aligned}$$

Let $w \in \{w_1, ..., w_k\}$ be the point where the maximum above is attained. Then

$$\frac{1}{n} \log \mathcal{Z}^F(n, \varepsilon) \leq \frac{\log k}{n} + F(w) + \frac{\eta}{3} + \frac{1}{n} \log \mathrm{card} E_n(\varepsilon, w, r).$$

Taking the appropriate limits as $n \to \infty$ and $\varepsilon \to 0$ we see that

$$\Pi^F_{\rm top}(\Phi) \leq F(w) + \frac{\eta}{3} + h(w, r). \tag{13}$$

Next we find a point $v \in \mathrm{Rot}_{\rm Pt}(\Phi)$ close to w such that $h(v) \geq h(w, r) - \frac{\eta}{3}$. Since $\overline{D(r, w)}$ is compact, it admits a finite cover by open balls of radius $\frac{r}{2}$ centered at some of its points. Let $u_1, ..., u_l \in \overline{D(r, w)}$ be the centers of those balls, then $\overline{D(r, w)} \subset \cup_{i=1}^{l} D(\frac{r}{2}, u_i)$ and

$$\mathrm{card} E_n(\varepsilon, w, r) \leq \sum_{i=1}^{l} \mathrm{card} E_n(\varepsilon, u_i, \tfrac{r}{2}).$$

Pick $v_1 \in \{u_1, ..., u_l\}$ such that $\mathrm{card} E_n(\varepsilon, v_1, \frac{r}{2}) = \max\limits_{1 \leq i \leq l} \mathrm{card} E_n(\varepsilon, u_i, \frac{r}{2})$. Then $\mathrm{card} E_n(\varepsilon, w, r) \leq l \, \mathrm{card} E_n(\varepsilon, v_1, \frac{r}{2})$, which implies that

$$\begin{aligned}
h(w, r) &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{card} E_n(\varepsilon, w, r) \\
&\leq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log l \, \mathrm{card} E_n(\varepsilon, v_1, \tfrac{r}{2}) \\
&= h(v_1, \tfrac{r}{2}).
\end{aligned}$$

We repeat this procedure to find $v_2 \in \overline{D(\frac{r}{2}, v_1)}$ such that $h(v_1, \frac{r}{2}) \leq h(v_2, \frac{r}{4})$, and so on. We iteratively construct a sequence of points $(v_k)_{k=1}^{\infty}$ such that

$\|v_k - v_{k+1}\| \le \frac{r}{2^k}$ and $h(v_{k-1}, \frac{r}{2^{k-1}}) \le h(v_k, \frac{r}{2^k})$ for all $k \in \mathbb{N}$. Note that (v_k) is Cauchy and hence converges. Denote $v = \lim_{k\to\infty} v_k$. Since

$$\|w - v\| \le \|w - v_1\| + \sum_{k=1}^{\infty} \|v_k - v_{k+1}\| \le 2r,$$

it follows fom the uniform continuity of F that $|F(w) - F(v)| < \frac{\eta}{3}$. On the other hand, $h(v) = \lim_{\rho\to 0} h(v, \rho)$. So for small enough ρ and large enough k we have

$$h(v) > h(v, \rho) - \frac{\eta}{3} > h(v_k, \tfrac{r}{2^k}) - \frac{\eta}{3} \ge h(w, r) - \frac{\eta}{3}.$$

Combining the last two observations with (13) we obtain $\Pi^F_{\text{top}}(\Phi) \le F(v) + h(v) + \eta$. Hence,

$$\Pi^F_{\text{top}}(\Phi) \le \sup_{w\in \mathrm{Rot}_{Pt}(\Phi)} \{h(w) + F(w)\}$$

and the proof is complete.

□

Remark 1. *Since* $\mathrm{Rot}_{Pt}(\Phi)$ *is compact, the map* $w \mapsto h(w)$ *is upper semi-continuous and* $F(w)$ *is continuous, the supremum in Theorem 1 is actually attained, i.e. there is* $w \in \mathrm{Rot}_{Pt}(\Phi)$ *such that* $\Pi^F_{\text{top}}(\Phi) = h(w) + F(w)$.

In [2] Buzzi, Kloeckner, and Leplaideur generalize the classical Varitional Principle to the nonlinear topological pressure. They show that for any continuous nonlinearity $F : \mathbb{R}^m \to \mathbb{R}$ and any continuous potentials $\phi_i : X \to \mathbb{R}$, $i = 1, ..., m$ we have

$$\Pi^F_{\text{top}}(\phi) = \sup_{\mu\in\mathcal{M}} \Pi^F(T, \phi_1, ..., \phi_m, \mu), \tag{14}$$

provided that there is an *abundance of ergodic measures*. The last property can be interpreted as a sort of irreducibility statement and is defined as follows. The system $(T, \phi_1, ..., \phi_m)$ has an abundance of ergodic measures if for any $\mu \in \mathcal{M}$ and $\varepsilon > 0$ there is $\nu \in \mathcal{M}^e$ such that $|h_\mu - h_\nu| < \varepsilon$ and $|\int \phi_i \, d\mu - \int \phi_i \, d\nu| < \varepsilon$ for $i = 1, ..., m$. It was noted in [2] that this property is essential for the nonlinear Variational Principle to hold. Indeed, one can take a dynamical system (X, T) consisting of just two fixed points and easily conjure up a potential ϕ and a continuous F on $\mathbb{R}$ such that the equality in (14) fails.

We provide an alternative proof of (14). Rather than following the approach of [2], we make use of Theorem 1 and rely on the results from the rotation theory. Using the terminology of rotation vectors we can rewrite (14) as

$$\Pi^F_{\text{top}}(\phi) = \sup_{w\in \mathrm{Rot}(\phi)} \left\{ \sup\{h_\mu : \mu \in \mathcal{M} \text{ and } \mathrm{rv}_\phi(\mu) = w\} + F(w) \right\},$$

and observe that the first component of the sum corresponds to the measure-theoretic entropy function which was introduced by Geller and Misiurewicz back in the nineties [3].

We now fix a multidimensional continuous potential $\Phi : X \to \mathbb{R}^m$ and a continuous nonlinearity $F : \mathbb{R}^m \to \mathbb{R}$. For $w \in \mathbb{R}^m$ we call $\mathcal{M}_\Phi(w) = \{\mu \in \mathcal{M} : \mathrm{rv}_\Phi(\mu) = w\}$ the rotation class of w. A localized variational principle for entropy proven in [11] and [10] states that for any $w \in \mathrm{Rot}_{Pt}(\Phi)$ we have

$$h(w) = \sup\{h_\mu : \mu \in \mathcal{M}_\Phi(w)\} \tag{15}$$

provided that the following two conditions are satisfied:

(i) there exists a sequence $(\mu_n) \subset \mathcal{M}^e$ such that $\mathrm{rv}_\Phi(\mu_n) \to w$ and $h_{\mu_n} \to \sup\{h_\mu : \mu \in \mathcal{M}_\Phi(w)\}$ (*approximation of the measure-theoretic entropy at w by ergodic measures*);

(ii) the function $v \mapsto \sup\{h_\mu : \mu \in \mathcal{M}_\Phi(v)\}$ is continuous at w (*continuity of the measure-theoretic entropy function at w*).

Since combining equality (15) and Theorem 1 immediately gives the multidimensional analog of (14), we need to examine conditions (i) and (ii) and determine their feasibility to hold for *every* $w \in \mathrm{Rot}(\Phi)$.

Clearly, for the nonlinear variational principle to hold we need to assume that the system (T, Φ) has an abundance of ergodic measures, i.e. for any invariant measure μ and $\varepsilon > 0$ there is an ergodic measure ν such that $|h_\mu - h_\nu| < \varepsilon$ and $\|\mathrm{rv}_\Phi(\mu) - \mathrm{rv}_\Phi(\nu)\| < \varepsilon$. This takes care of the condition (i) in the localized variational principle (15), since now the measure-theoretic entropy can be approximated by ergodic measures at any point. Moreover, it follows from the Birkhoff ergodic theorem that for such systems we have $\mathrm{Rot}_{Pt}(\Phi) = \mathrm{Rot}(\Phi)$.

We turn our attention to the condition (ii). Since the entropy map $\mu \to h_\mu$ is affine, $\sup\{h_\mu : \mu \in \mathcal{M}_\Phi(v)\}$ is a concave function of v. Together with the fact that the rotation set is convex, this implies that the function $v \mapsto \sup\{h_\mu : \mu \in \mathcal{M}_\Phi(v)\}$ is continuous in the interior of $\mathrm{Rot}(\Phi)$ (see e.g. [17]). Unfortunately, the continuity does not extend to the boundary even under additional (natural) assumptions on (T, Φ). Indeed, in [19] Wolf constructed a 2-dimensional Lipschitz potential Φ on a full shift, for which the measure-theoretic entropy function is discontinuous at a boundary point.

Hence, the localized variational principle may not hold when w is a boundary point of $\mathrm{Rot}(\Phi)$, and a specific example of such occurrence is provided in [12]. The good news is that we do not need the equality in (15) to take place for every w, since we are interested in the supremum over all points in the rotation set. We have to seek a relation between $h(w)$ and $\sup\{h_\mu : \mu \in \mathcal{M}_\Phi(v)\}$ which does not rely on the continuity of the latter. In the next lemma we modify the proof of the localised variational principle (15) to remove condition (ii); in fact, we remove condition (i) as well. We obtain a weaker conclusion, but it is enough for our purposes.

Lemma 1. *Let $T : X \to X$ be a continuous map on a compact metric space X and $\Phi : X \to \mathbb{R}^m$ be a continuous potential. Then for every $w \in \mathrm{Rot}_{Pt}(\Phi)$ we have*

$$\limsup_{v\to w} \sup \{h_\mu : \mu \in \mathcal{M}_\Phi(v) \cap \mathcal{M}^e\} \le h(w) \le \limsup_{v\to w} \sup \{h_\mu : \mu \in \mathcal{M}_\Phi(v)\}$$

Proof. The reason for the lower bound to take place is that the entropy of an ergodic measure can be defined in a similar manner to the topological entropy of T through optimal covers by balls in the d_n-metric. The difference is that we take covers of some subsets of positive measure instead of the whole space X. This useful feature of ergodic measures was discovered by Katok in [8]. Precisely, for $\mu \in \mathcal{M}^e$, $\delta > 0$ and $\varepsilon > 0$ we denote by $N_\mu(n, \varepsilon, \delta)$ the minimal number of ε-balls in the d_n-metric which cover the set of μ-measure more than or equal to $1 - \delta$. Then for any $\delta > 0$

$$h_\mu = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log N_\mu(n, \varepsilon, \delta). \tag{16}$$

To establish the first inequality of the lemma it is enough to show that for any $\eta > 0$ there is $r > 0$ such that any $\mu \in \mathcal{M}^e$ with $\mathrm{rv}_\Phi(\mu) \in D(r, w)$ has entropy $h_\mu \le h(w) + \eta$. For every $\varepsilon > 0$ and every $n \in \mathbb{N}$ we select a finite cover of X by ε-balls in the d_n-metric and denote by $J_n(\varepsilon)$ the set of their centers. From (9) we can find $r > 0$ for which $h(w, 3r) < h(w) + \eta$. Let $\mu \in \mathcal{M}^e$ be such that $\mathrm{rv}_\Phi(\mu) \in D(w, r)$. It follows from Birkhoff Ergodic Theorem that $\frac{1}{n} S_n\Phi(x) \to \mathrm{rv}_\Phi(\mu)$ for μ-almost all $x \in X$. Therefore,

$$A_n = \left\{ x \in X : \left\| \frac{1}{k} S_k\Phi(x) - \mathrm{rv}_\Phi(\mu) \right\| < r \text{ for all } k > n \right\} \tag{17}$$

form a sequence of nested sets with $\mu(\cup A_n) = 1$. We are only interested in those balls in the coverings of X which intersect A_n, i.e. we consider $J'_n(\varepsilon) = \{x \in J_n(\varepsilon) : B_n(x, \varepsilon) \cap A_n \ne \varnothing\}$. Uniform continuity of Φ implies that for some sufficiently small ε_0 we have $\|\frac{1}{n} S_n\Phi(x) - \frac{1}{n} S_n\Phi(y)\| < r$ whenever $d_n(x, y) < \varepsilon_0$. Then for $\varepsilon < \varepsilon_0$ and $x \in J'_n(\varepsilon)$ the Birkhoff average $\frac{1}{n} S_n\Phi(x) \in D(w, 3r)$. Indeed, we can take $y \in B_n(x, \varepsilon) \cap A_n$ and use (17) to estimate

$$\begin{aligned} \|\tfrac{1}{n} S_n\Phi(x) - w\| \le \|\tfrac{1}{n} S_n\Phi(x) - \tfrac{1}{n} S_n\Phi(y)\| + \|\tfrac{1}{n} S_n\Phi(y) - \mathrm{rv}_\Phi(\mu)\| \\ + \|\mathrm{rv}_\Phi(\mu) - w\| \\ < 3r. \end{aligned}$$

We pick some $0 < \delta < 1$ and note that since $\mu(A_n) \to 1$ there is $n_0 \in \mathbb{N}$ such that for $n > n_0$ we have $\mu\left(\cup_{x\in J'_n(\varepsilon)} B_n(x, \varepsilon)\right) > 1 - \delta$ and hence $N_\mu(n, \varepsilon, \delta) \le \mathrm{card} J'_n(\varepsilon)$. We filter the set $J'_n(\varepsilon)$ even further. We apply Vitali covering lemma to find a subset $J''_n(\varepsilon) \subset J'_n(\varepsilon)$ such that the balls $B_n(x, \varepsilon)$ with $x \in J''_n(\varepsilon)$ are pairwise disjoint and

$$\bigcup_{x\in J'_n(\varepsilon)} B_n(x, \varepsilon) \subset \bigcup_{x\in J''_n(\varepsilon)} B_n(x, 3\varepsilon).$$

By construction, for $\varepsilon < \varepsilon_0$ and $n > n_0$ the set $J''_n(\varepsilon)$ is (n,ε)-separated, $\frac{1}{n}S_n\Phi(x) \in D(w,3r)$ whenever $x \in J''_n(\varepsilon)$, and $\mu\left(\cup_{x\in J''_n(\varepsilon)} B_n(x,3\varepsilon)\right) > 1-\delta$. Therefore, $N_\mu(n,3\varepsilon,\delta) \le \mathrm{card} J''_n(\varepsilon) \le \mathrm{card} E_n(\varepsilon,w,3r)$ where $E_n(\varepsilon,w,3r)$ are the sets used in the definition of the localized topological entropy (8). Using Katok's formula (16) and (8) we get

$$\begin{aligned} h_\mu &= \lim_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log N_\mu(n,3\varepsilon,\delta) \\ &\le \lim_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log E_n(\varepsilon,w,3r) \\ &\le \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log E_n(\varepsilon,w,3r) \\ &= h(w,3r). \end{aligned} \tag{18}$$

Since r was chosen such that $h(w,3r) < h(w)+\eta$, we reach the desired conclusion.

To proof the upper bound on the localized topological entropy we utilize the standard technique of Misiurewicz to construct measures with large entropy using atomic measures concentrated on (n,ε)-separated sets (see [18] or [9, Section 4.5]). Since $\sup\{h_\mu : \mu\in\mathcal{M}_\Phi(v)\}$ is a concave function of v we do not need to keep track of its value at w. It is suffices to show that for any $\eta > 0$ and $r > 0$ there is $\mu \in \mathcal{M}$ such that $\mathrm{rv}_\Phi(\mu) \in D(w,r)$ and $h(w) - \eta \le h_\mu$. For fixed η and r we find $\varepsilon > 0$ such that

$$h(w) \le h(w,r) \le \limsup_{n\to\infty}\frac{1}{n}\log \mathrm{card} E_n(\varepsilon,w,r) + \eta. \tag{19}$$

Let μ be an accumulation point of sequence of probability measures (μ_n) where

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n-1}\nu_n\circ T^{-k} \quad \text{with} \quad \nu_n = \frac{1}{\mathrm{card}E_n(\varepsilon,w,r)}\sum_{x\in E_n(\varepsilon,w,r)}\delta_x.$$

Then μ is T-invariant and its entropy $h_\mu \ge \limsup_{n\to\infty}\frac{1}{n}\log\mathrm{card}E_n(\varepsilon,w,r)$. Combining this with (19) completes the proof. □

We are ready to establish the Nonlinear Variational Principle, which now follows immediately from Theorem 1 and Lemma 1.

Theorem 2. *Let $T : X \to X$ be a continuous map on a compact metric space X and $F : \mathbb{R}^m \to \mathbb{R}$ be a continuous nonlinearity. Suppose $\Phi : X \to \mathbb{R}^m$ is a continuous potential such that the system (T,Φ) has an abundance of ergodic measures. Then*

$$\Pi^F_{\mathrm{top}}(\Phi) = \sup_{\mu\in\mathcal{M}}\{h_\mu + F(\mathrm{rv}_\Phi(\mu))\}$$

Proof. According to Lemma 1, for any $w \in \mathrm{Rot}_{Pt}(\Phi)$ we have

$$\begin{aligned} h(w) + F(w) &\leq \limsup_{v \to w} \Big(\sup\{h_\mu : \mu \in \mathcal{M}_\Phi(v)\} + F(v) \Big) \\ &= \limsup_{v \to w} \sup_{\mu \in \mathcal{M}_\Phi(v)} \{h_\mu + F(\mathrm{rv}_\Phi(\mu))\} \\ &\leq \sup_{\mu \in \mathcal{M}} \{h_\mu + F(\mathrm{rv}_\Phi(\mu))\} \end{aligned}$$

We apply Theorem 1 and see that

$$\Pi^F_{\mathrm{top}}(\Phi) \leq \sup_{\mu \in \mathcal{M}} \{h_\mu + F(\mathrm{rv}_\Phi(\mu))\}. \tag{20}$$

To obtain the opposite inequality, consider a sequence of ergodic measures (μ_n) which approximates $\sup_{\mu \in \mathcal{M}^e} \{h_\mu + F(\mathrm{rv}_\Phi(\mu))\}$. Let $\mu \in \mathcal{M}$ be an accumulation point of (μ_n) and $w = \mathrm{rv}_\Phi(\mu)$. It follows from the Birchoff Ergodic Theorem that $w \in \mathrm{Rot}_{Pt}(\Phi)$. Therefore, we can use Lemma 1 and get

$$\begin{aligned} \sup_{\mu \in \mathcal{M}^e} \{h_\mu + F(\mathrm{rv}_\Phi(\mu))\} &= \lim_{n \to \infty} (h_{\mu_n} + F(\mathrm{rv}_\Phi(\mu_n))) \\ &\leq \limsup_{v \to w} \Big(\sup\{h_\mu : \mu \in \mathcal{M}_\Phi(v) \cap \mathcal{M}^e\} + F(v) \Big) \\ &\leq h(w) + F(w) \\ &\leq \Pi^F_{\mathrm{top}}(\Phi). \end{aligned} \tag{21}$$

Since (T, Φ) has an abundance of ergodic measures, for any $\mu \in \mathcal{M}$ we can find a sequence of ergodic measures (μ_n) such that $h_{\mu_n} \to h_\mu$ and $\mathrm{rv}_\Phi(\mu_n) \to \mathrm{rv}_\Phi(\mu_n)$. The continuity of F assures that $h_{\mu_n} + F(\mathrm{rv}_\Phi(\mu_n)) \to h_\mu + F(\mathrm{rv}_\Phi(\mu))$ and hence, the supremum on the left hand side of (21) coincides with the supremum on the right hand side of (20). □

Remark 2.
(1) For any compact dynamical system (X, T), continuous potential $\Phi : X \to \mathbb{R}^m$ and continuous nonlinearity $F : \mathbb{R}^m \to \mathbb{R}$ we have the inequalities

$$\sup_{\mu \in \mathcal{M}^e} \{h_\mu + F(\mathrm{rv}_\Phi(\mu))\} \leq \Pi^F_{\mathrm{top}}(\Phi) \leq \sup_{\mu \in \mathcal{M}} \{h_\mu + F(\mathrm{rv}_\Phi(\mu))\}. \tag{22}$$

The assumption of an abundance of ergodic measures is used to guarantee that the supremum on the right hand side coincides with the supremum on the left hand side.

(2) Another instance when the suprema in (22) are the same is when the nonlinearity map $F : \mathbb{R}^m \to \mathbb{R}$ is convex. As was pointed out in [2], *this follows immediately from the Ergodic Decomposition Theorem and Jensen's inequality. Indeed, for $\mu \in \mathcal{M}$ we write $\mu = \int \mu_x d\mu(x)$*

for the ergodic decomposition of μ. *Then*

$$\begin{aligned} h_\mu + F(\mathrm{rv}_\Phi(\mu)) &= \int h_{\mu_x} d\mu(x) + F\left(\int \mathrm{rv}_\Phi(\mu_x) d\mu(x)\right) \\ &\leq \int \Big(h_{\mu_x} + F(\mathrm{rv}_\Phi(\mu_x))\Big) d\mu(x) \\ &\leq \sup_{\nu\in\mathcal{M}^e} \{h_\nu + F(\mathrm{rv}_\Phi(v))\}. \end{aligned}$$

(3) *If the system* (T,Φ) *has an abundance of ergodic measures then* $\mathrm{Rot}_{Pt}(\Phi) = \mathrm{Rot}(\Phi)$. *In view of Theorem 1 one might wonder whether the equality of rotation sets along implies the nonlinear variational principle. This is not true, as evident from Example 1 and the discussion afterwards.*

In [2] the authors provide a simple example to show that the inequality on the right hand side of (22) can be strict. They consider a system (X,T) which is a union of two distinct fixed points p and q and define a potential $\phi : X \to \mathbb{R}$ as $\phi(p) = -1$, $\phi(q) = 1$. Taking the quadratic nonlinearity $F(w) = -w^2$ they obtain that $\Pi^F_{\mathrm{top}}(\phi) = -1$ but $h_\mu + F(\mathrm{rv}_\phi(\mu)) = 0$ for $\mu = \frac{1}{2}(\delta_p + \delta_q)$. Motivated by this example and the one-dimensional analog of (22) the authors ask whether the nonlinear variational principle holds in restriction to ergodic measures [2, Question 1.6]. Precisely, without assuming abundance of ergodicity or convexity of the nonlinearity is it true that

$$\sup_{\mu\in\mathcal{M}^e} \{h_\mu + F(\mathrm{rv}_\phi(\mu))\} = \Pi^F_{\mathrm{top}}(\phi)?$$

The next example shows that the answer is no. Our inspiration once again comes from rotation theory. Note that in the construction above $\mathrm{Rot}_{Pt}(\phi) = \{-1, 1\}$ with both points being rotation vectors of ergodic measures, however $\mathrm{Rot}(\phi) = [-1, 1]$. Hence, the nonlinear pressure is computed using the values of F at the end points of $[-1, 1]$, but the whole interval is available to maximize the free energies. In the construction below we fill in the interior of $[-1, 1]$ with some elements of the pointwise rotation set while keeping the rotation vectors of ergodic measures the same. This will show that the left hand side inequality in (22) can be strict as well.

Example 1. *There exists a subshift* X *on two symbols, a locally constant potential* $\phi : X \to \mathbb{R}$ *and a nonlinearity* $F : \mathbb{R} \to \mathbb{R}$ *such that*

$$\sup_{\mu\in\mathcal{M}^e} \{h_\mu + F(\mathrm{rv}_\phi(\mu))\} < \Pi^F_{\mathrm{top}}(\phi)$$

Proof. Let (Σ_2, T) be the full double-sided shift on the alphabet $\{0, 1\}$. Consider a subshift $X \subset \Sigma_2$ given by

$$(x_j)_{j=-\infty}^{\infty} \in X \iff \text{there is } k \in \mathbb{Z} \text{ such that } x_j = x_{j+1} \text{ for all } j \neq k.$$

Hence, X consists of the sequences with at most one transition from zero to one or from one to zero. We define the potential $\phi : X \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} -1, & \text{if } x_0 = 0; \\ 1, & \text{if } x_0 = 1, \end{cases}$$

and the nonlinearity $F : \mathbb{R} \to \mathbb{R}$ by $F(w) = -w^2$. Clearly, the only ergodic T-invariant measures on X are the point mass measures at $(\bar{0})$ and $(\bar{1})$. Therefore,

$$\sup_{\mu \in \mathcal{M}^e} \{h_\mu + F(\mathrm{rv}_\phi(\mu))\} = -1.$$

On the other hand, we can check that $\Pi^F_{\mathrm{top}}(\phi) > -1$. For any $k \in \mathbb{N}$ take element $x^{(k)} = (x^{(k)}_j)_{j=-\infty}^{\infty}$ of X which has transition from zero to one at k, i.e.

$$x^{(k)}_j = \begin{cases} 0, & \text{if } j < k; \\ 1, & \text{if } j \geq k. \end{cases} \tag{23}$$

Then for $n = 2k$ we have $\frac{1}{n} S_n \phi(x^{(k)}) = 0$. It follows directly from the definition of the pointwise rotation set (6) that $0 \in \mathrm{Rot}_{Pt}(\phi)$. Given that the topological entropy of the system is zero and $F(w)$ attains its maximum at $w = 0$ we see that $\Pi^F_{\mathrm{top}}(\phi) = 0$ by Theorem 1. □

In the above example we have $\Pi^F_{\mathrm{top}}(\phi) = \sup_{\mu \in \mathcal{M}} \{h_\mu + F(\mathrm{rv}_\phi(\mu))\}$. However, we can easily modify it so that the supremum on the right hand side becomes larger. We just need to increase the entropies of the end points of $[-1, 1]$, which we accomplish by replacing $(\bar{0})$ and $(\bar{1})$ with additional full shifts on alphabets $\{0, 2\}$ and $\{1, 3\}$ respectively. We take $Y = X \cup \{0, 2\}^{\mathbb{Z}} \cup \{1, 3\}^{\mathbb{Z}}$, where X is the subshift from Example 1. We extend the potential ϕ from X to Y by letting $\phi \equiv -1$ on $\{0, 2\}^{\mathbb{Z}}$ and $\phi \equiv 1$ on $\{1, 3\}^{\mathbb{Z}}$ and consider the nonlinearity $F(w) = (-2 \log 2) w^2$. All ergodic T-invariant measures are supported either on $\{0, 2\}^{\mathbb{Z}}$ or on $\{1, 3\}^{\mathbb{Z}}$ and hence

$$\sup_{\mu \in \mathcal{M}^e} \{h_\mu + F(\mathrm{rv}_\phi(\mu))\} = -\log 2.$$

Now consider $w \in (-1, 1)$. Note that all the elements of Y whose Birkhoff averages are sufficiently close to w must belong to X. Therefore, $h(w) = 0$ for all $w \in \mathrm{Rot}_{Pt}(\phi)$ with exception of $w = -1$ and $w = -1$. It follows from Theorem 1 that $\Pi^F_{\mathrm{top}}(\phi) = 0$. At the same time,

$$\sup_{\mu \in \mathcal{M}} \{h_\mu + F(\mathrm{rv}_\phi(\mu))\} = \log 2,$$

since the maximums of $h(w)$ and $F(w)$ are both realized at the rotation vector of $\mu = \frac{1}{2}(\mu_1 + \mu_2)$, where μ_1 and μ_2 are the measures of maximal entropy of $\{0, 2\}^{\mathbb{Z}}$ and $\{1, 3\}^{\mathbb{Z}}$ respectively.

Furthermore, the pointwise rotation set for the potential in Example 1 (as well as its current modification) makes up the whole interval $[-1, 1]$. To see this, let $\frac{p}{q}$ be any proper positive fraction. For $l \in \mathbb{N}$ consider $x^{(k)} \in X$ as defined in (23) with $k = l(q - p)$ and take $n = 2lq + 1$. Then

$\frac{1}{n}S_n\phi(x^{(k)}) = \frac{2lp+1}{2lq+1}$, which can be made arbitrarily close to $\frac{p}{q}$ with large enough l. Therefore, $\frac{p}{q} \in \mathrm{Rot}_{Pt}(\phi)$. Similar argument works for negative fractions, we just need to consider elements of X for which the transition at k is from one to zero. The compactness of the pointwise rotation set now implies that $\mathrm{Rot}_{Pt}(\phi) = [-1, 1]$. As a result, we have a situation where both inequalities of (22) are strict while the pointwise and generalized rotation sets coincide.

It is well known that in the definition of classical pressure (3) one can use the lower limit as n tends to infinity instead of the upper limit [9, Section 20.2]. Surprisingly, this is in general not true for the nonlinear pressure, as we show in Example 2 below. However, we can still replace the upper limit in (11) by the lower limit under the hypothesis of Theorem 2.

Corollary 1. *Let $T : X \to X$ be a continuous map on a compact metric space X and $F : \mathbb{R}^m \to \mathbb{R}$ be a continuous nonlinearity. Suppose $\Phi : X \to \mathbb{R}^m$ is a continuous potential such that the system (T, Φ) has an abundance of ergodic measures. Then*

$$\Pi^F_{\mathrm{top}}(\Phi) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \mathcal{Z}^F(n, \varepsilon). \tag{24}$$

Proof. For the purpose of this corollary we define the lower localized entropy as

$$\underline{h}(w) = \lim_{r\to 0} \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \mathrm{card} E_n(\varepsilon, w, r),$$

where $E_n(\varepsilon, w, r)$ are as in (8). Then taking the lower limit rather than the upper limit in (12) we obtain that

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \mathcal{Z}^F(n, \varepsilon) \geq \sup_{w\in \mathrm{Rot}_{Pt}(\Phi)} \{\underline{h}(w) + F(w)\}.$$

We note that in (18) Katok's formula provides the lower limit in the estimate of h_μ, so the proof of Lemma 1 actually gives

$$\limsup_{v\to w} \sup \{h_\mu : \mu \in \mathcal{M}_\Phi(v) \cap \mathcal{M}^e\} \leq \underline{h}(w).$$

Now we can repeat the argument in Theorem 2 and conclude that

$$\sup_{\mu\in\mathcal{M}^e} \{h_\mu + F(\mathrm{rv}_\Phi(\mu))\} \leq \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \mathcal{Z}^F(n, \varepsilon).$$

When there is an abundance of ergodic measures, the last inequality implies the assertion of the corollary. □

Remark 3. *The statement of the corollary remains true as long as the supremum on the left hand side of (22) coincides with the supremum on the right hand side, e.g. when F is convex (see Remark 2, part 2).*

We conclude this paper with a construction of an example which shows that in general one cannot replace the upper limit in the definition of the nonlinear pressure (11) by the lower limit. This contrasts with the classical thermodynamic formalism where we always have the equality of the corresponding quantities.

Example 2. *There exist a dynamical system (X,T), a continuous potential Φ on X, and a continuous nonlinearity F such that*

$$\lim_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\mathcal{Z}^F(n,\varepsilon)\neq\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\mathcal{Z}^F(n,\varepsilon).$$

Proof. Let (Σ_3,T) be a full double-sided shift on the alphabet $\{0,1,2\}$. Consider a subshift $X\subset\Sigma_3$ given by

$$(x_j)_{j=-\infty}^{\infty}\in X \iff \text{there are } k\in\mathbb{Z} \text{ and } m\in\mathbb{N} \text{ such that } x_j=x_{j+1} \text{ for all } j\notin\{k,k+5^m\}.$$

Hence, X consists of the sequences with at most two transitions between the elements of the alphabet. Moreover, if a sequence does have two transitions, the length of the middle block must be a power of 5. It is easy to check that the topological entropy of X is zero.

Pick any three points w_0,w_1 and w_2 in $\mathbb{R}^2$ which form an equilateral triangle and denote by c its center. Define the potential $\Phi:X\to\mathbb{R}^2$ by $\Phi(x)=w_i$ if $x_0=i\,(i=0,1,2)$ and the nonlinearity $F:\mathbb{R}^2\to\mathbb{R}$ by $F(w)=-\|w-c\|^2$. For $m\in\mathbb{N}$ let $x^{(m)}\in X$ be such that

$$x_j^{(m)}=\begin{cases}0, & \text{if } j<5^m;\\ 1, & \text{if } 5^m\le j<5^{m+1};\\ 2, & \text{if } j\ge 5^{m+1},\end{cases}$$

Taking $n=3\cdot 5^m$ we obtain that $\frac{1}{n}S_n\Phi(x^{(m)})=c$. Consequently $c\in\mathrm{Rot}_{Pt}(\Phi)$ and $\Pi^F_{\text{top}}(\Phi)=0$.

Consider sequence $n_m=5^m-1$. For any $x\in X$ we estimate the value of $F\left(\frac{1}{n_m}S_{n_m}\Phi(x)\right)$. If x has fewer than two transitions, then its Birkhoff average lies on the line connecting two of the points w_i. If we denote by s the side length of the triangle with vertices at w_i, then for such x we have $F(x)\le\frac{-s^2}{12}$. Meanwhile, if x has two transitions within the first n_m coordinates, then the maximum length of the middle block is 5^{m-1}. We can write a point w of the form $\frac{1}{n_m}S_{n_m}\Phi(x)$ as $w=p_0w_0+p_1w_1+p_2w_2$ where nonnegative coefficients p_i satisfy $p_0+p_1+p_2=0$. In addition, for the coefficient corresponding to the middle block, say p_0, we have $p_0\le\frac{5^{m-1}}{5^m-1}\le\frac{1}{4}$. A convenient way to compute the distance between point w and the center of the triangle c is by employing barycentric coordinates. The triple (p_0,p_1,p_2) is the barycentric coordinates of w, and the center c has coordinates $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$. Hence,

$$-\|c-w\|=s^2\left[(p_0-\tfrac{1}{3})(p_1-\tfrac{1}{3})+(p_1-\tfrac{1}{3})(p_2-\tfrac{1}{3})+(p_2-\tfrac{1}{3})(p_0-\tfrac{1}{3})\right].$$

It is an easy application of constraint optimization technique to check that the function above has maximum at $(\frac{1}{4},\frac{3}{8},\frac{3}{8})$. Therefore, for any $m\in\mathbb{N}$ and any $x \in X$

$$F\left(\frac{1}{n_m} S_{n_m}\Phi(x)\right) \leq -\frac{s^2}{192},$$

which implies that

$$\lim_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log \mathcal{Z}^F(n,\varepsilon) \leq \lim_{\varepsilon\to 0}\lim_{m\to\infty}\frac{1}{n_m}\log \mathcal{Z}^F(n_m,\varepsilon) \leq -\frac{s^2}{192}$$

whereas

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log \mathcal{Z}^F(n,\varepsilon) = \Pi^F_{\mathrm{top}}(\Phi) = 0$$

□

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